Warmups: (1) Find the points on $y = x^2 + 5$ and $y = 2x - 7$ that are closest. What is the distance?

[Do this 3 different ways.]

(2) Find $\int_0^1 \int_{x^2}^{x} \sin(x+2y)\, dy\, dx$,

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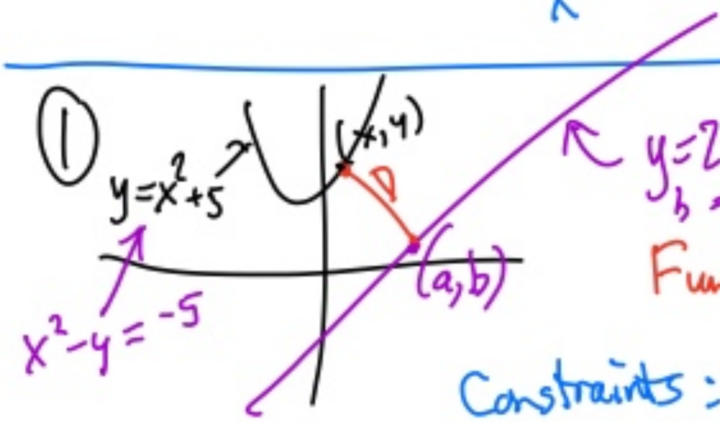
(1)



$\leftarrow y = 2x - 7$
$b = 2a - 7$

$F(x,y,a,b)$

Function: $F = D^2 = (x-a)^2 + (y-b)^2$

Constraints: $g_1(x,y,a,b) = x^2 - y = -5$
$g_2(x,y,a,b) = b - 2a = -7$

Equations: $\begin{cases} \nabla F = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 & \leftarrow \text{4 eqns} \\ g_1 = -5 \\ g_2 = -7 \end{cases}$ $\leftarrow$ 2 more equations

6 unknowns: $x, y, a, b, \lambda_1, \lambda_2$.

$F_x = 2(x-a) = \lambda_1(2x) + \lambda_2(0) = 2x\lambda_1$

$F_y = 2(y-b) = \lambda_1(-1) + \lambda_2(0) = -\lambda_1$

$F_a = 2(x-a)(-1) = \lambda_1(0) + \lambda_2(-2) = -2\lambda_2$

$F_b = 2(y-b)(-1) = \lambda_1(0) + \lambda_2(1) = \lambda_2$

$x^2 - y = -5$

$b - 2a = -7$

$\lambda_2 = -2y+2b = x-a$

$\lambda_1 = -2y+2b = 2\frac{(x-a)}{2x}$ ← $x \neq 0$

If $x=0$
$a=0$

$\Rightarrow \boxed{x=1}$!

$1-y=-5 \Rightarrow \boxed{6=y}$

$-12+2b=1-a$

$-12+2b=1-a$

$a+2b=13$

$b-2a=-7 \Rightarrow b=2a-7$

$\Rightarrow a+2(2a-7)=13$

$a+4a-14=13$

$5a=27$

$a=\left(\frac{27}{5}\right)$

$b=2a-7=2\left(\frac{27}{5}\right)-7$

$\frac{54}{5}-\frac{35}{5}=\frac{19}{5}$

$x=0=a$

$\lambda_2=0$

$y=b$

$y=5=b$

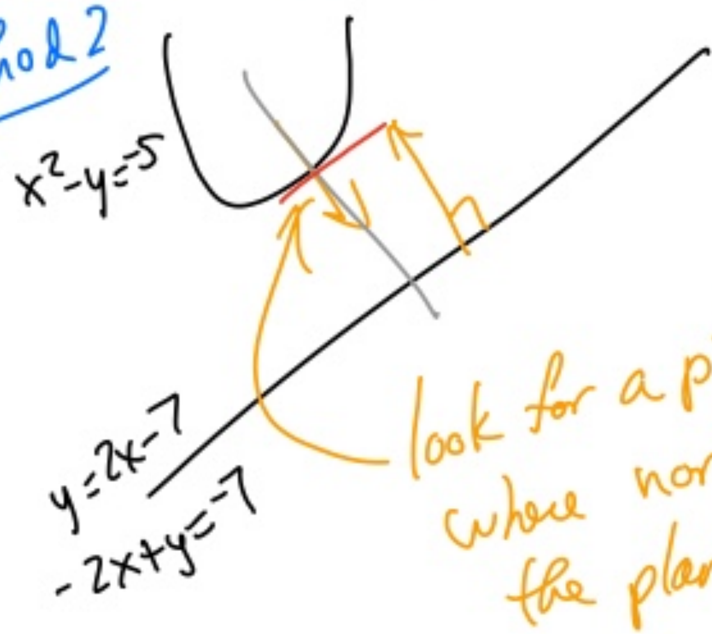
$b-2a=-7$

$5-0=-7$ impossible

$(x,y)=(1,6)$
$(a,b)=\left(\frac{27}{5},\frac{19}{5}\right)$

$D=\sqrt{\left(1-\frac{27}{5}\right)^2+\left(6-\frac{19}{5}\right)^2}$

Method 2



look for a pt on parabola whose normal is $\|$ to the plane normal

$$\Longleftrightarrow \nabla(x^2-y) = \lambda \nabla(-2x+y)$$
$$x^2-y=-5$$

$$\boxed{\begin{aligned} 2x &= \lambda(-2) \\ -1 &= \lambda(1) \\ x^2-y &= -5 \end{aligned}}$$

$\Rightarrow \lambda = -1$

$x = 1$

$y = 6$

$$(x,y) = (1,6)$$

Normal vector – $(2x,-1) = \underset{\Delta x \ \Delta y}{(2,-1)}$

$\left(\begin{matrix}\text{slope} \\ \text{of perp line}\end{matrix}\right) = \frac{-1}{2}$

$$(y-6) = \frac{-1}{2}(x-1) \quad \text{equation of}$$
$$y-6 = -\frac{1}{2}x + \frac{1}{2}$$
$$2y-12 = -x+1$$
$$x+2y = +13$$

$$-2x + y = -7$$

$$2x + 4y = +26$$

$$5y = \cancel{-33}\ 19$$

$$y = \frac{\cancel{-33}\ 19}{5}$$ ← on line.

$$x = +13 - 2y = \boxed{+13 - \frac{38}{5}} = \frac{27}{5}$$

$$\begin{array}{r} 65 \\ 38 \\ \hline 27 \end{array}$$

$$(x,y) = (1,6) \qquad \left(\frac{27}{5}, \frac{19}{5}\right) = (a,b)$$

hmm – should be same

It is.

Method 3

$$\int_0^1 \int_{x^2}^{x} \sin(x+2y)\,dy\,dx$$

$x^2 - y = -5$

$y = x^2 + 5$

$y = 2x - 7$

find pt when slope = 2

$$y' = 2x = 2 \Rightarrow x = 1.$$

$$y = x^2 + 5 = 6.$$

$$(1,6)$$

slope of $\perp = -\frac{1}{2}$ (neg reciprocal)

$(y-6) = -\frac{1}{2}(x-1)$ intersect w/ $y = 2x-7$

$\Rightarrow (x,y) = \left(\frac{27}{5}, \frac{19}{5}\right)$ pt. on line.

$\circlearrowleft \sin(2y)$
$-\frac{1}{2}\cos(2y)$

② Find $\displaystyle \underbrace{\int_0^1 \int_{x^2}^{x} \sin(x+2y)\,dy}_{1\text{st}}\,dx$,

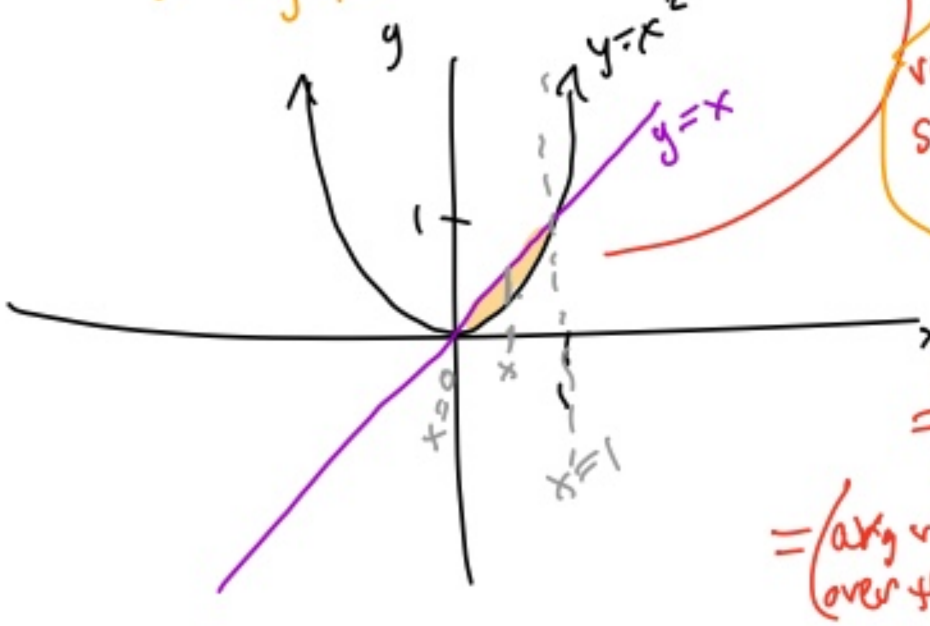
$$= \int_{x=0}^{x=1} \left(\left(-\frac{1}{2}\cos(x+2y) \right) \Big|_{x^2}^{x} \right) dx \qquad \frac{\partial}{\partial y}\left(-\frac{1}{2}\cos(x+2y)\right) = \sin(x+2y)$$

$$= \int_{x=0}^{1} \Big(\underbrace{-\frac{1}{2}\cos(x+2x)}_{\cos(3x)} + \frac{1}{2}\cos(x+2x^2) \Big)\,dx$$

$\overset{\text{numerical}}{=}$ . . . - -

What are we integrating over??

$$\int_{x=0}^{x=1} \int_{y=x^2}^{y=x} \sin(x+2y)\,dy\,dx$$

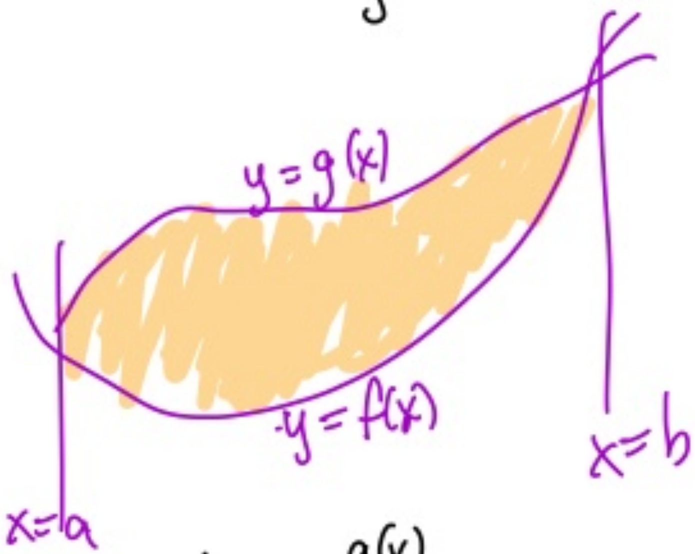


volume under surface $z = \sin(x+2y)$ over this region

$= \lim \sum \sin(x+2y)\,\Delta x\,\Delta y$

$=$ (avg value of $\sin(x+2y)$ over that region) $\cdot$ area of region.

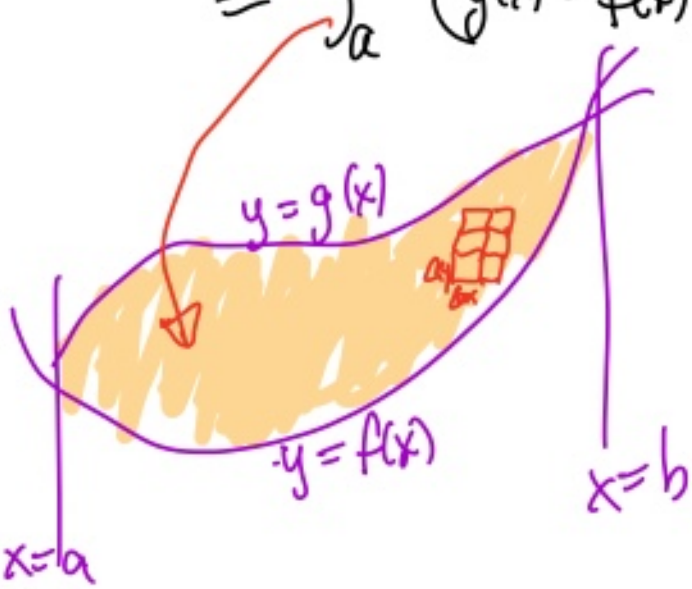
Example: $$\int_{x=a}^{b}\int_{y=f(x)}^{g(x)} \sim\sim\sim \; dy\, dx$$



Example $$\int_{x=a}^{b}\int_{y=f(x)}^{g(x)} 1 \; dy\, dx$$

$$= \int_a^b \left(y \Big|_{f(x)}^{g(x)} \right) dx$$

$$= \int_a^b \left(g(x) - f(x)\right) dx$$ = area between $y=f(x)$ & $y=g(x)$.

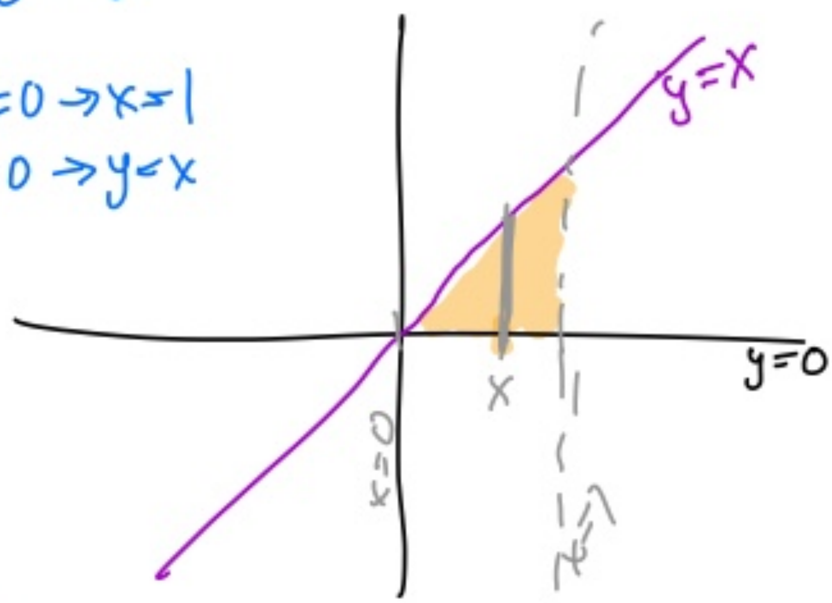


$\boxed{\text{Example}}$ – Different orders of integration

$f(x,y) = e^{x^2}$

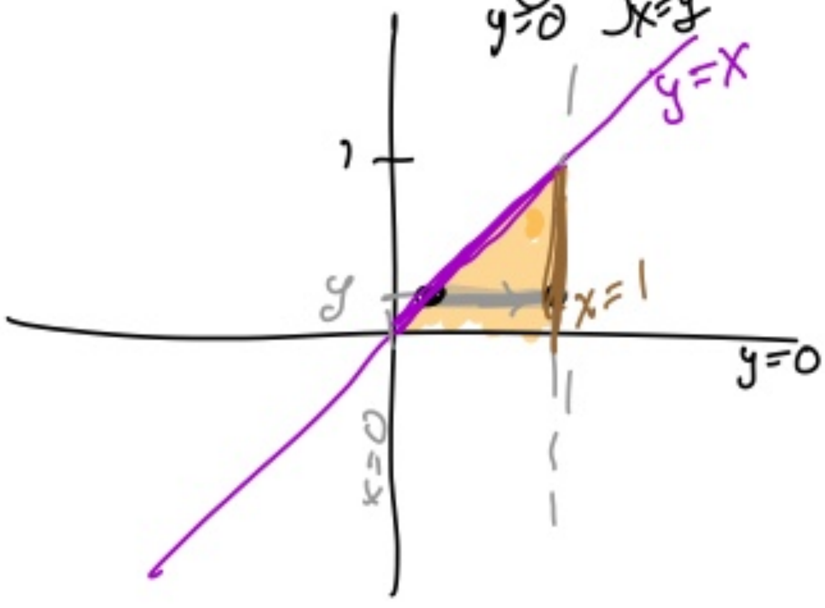
$$\int_0^1 \int_0^x e^{x^2}\, dy\, dx$$ ← integrating vertically first.

Picture: $x=0 \to x=1$
$y=0 \to y=x$



Same integral, different order:
integrate horizontally first (x, dx) then vertically.

$$\int_{y=0}^{y=1} \int_{x=y}^{x=1} e^{x^2}\, dx\, dy$$



We have (vertical first)

$$\int_0^1 \left(\int_0^x e^{x^2}\, dy \right) dx = \int_0^1 \left(\int_y^1 e^{x^2}\, dx \right) dy$$

can't do this one.

$$\begin{aligned}
&= \int_0^1 \left(y e^{x^2} \Big|_0^x \right) dx \\
&= \int_0^1 x e^{x^2}\, dx
\end{aligned}$$

$u = x^2$

$du = 2x\, dx$

$x\, dx = \frac{1}{2} du$

$$= \int_{u=0}^{u=1} \frac{1}{2} e^u\, du = \frac{1}{2} e^u \Big|_0^1 = \frac{e^1}{2} - \frac{e^0}{2}$$

$$= \boxed{\frac{e-1}{2}}.$$